Chapter

Fluid Motion Equations in Tensor Form

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Abstract

In the current chapter, some applications of tensor analysis to fluid dynamics are presented. Governing equations of fluid motion and energy are obtained and analyzed. We shall discuss about continuity equation, equation of motion, and mechanical energy transport equation and four forms of energy equation. Finally, we shall talk about the divergence from transfer equations of different parameters of motion. The tensor form of equations has advantages over the component form: these are, first, compact writing of equations and, second, independency from reference frames, etc. Moreover, it allows to obtain new forms of equations on the basis of governing ones easily.

Keywords: stress tensor, Navier–stokes equation, energy, continuity, vorticity, divergence form

1. Introduction

The mathematical model of moving fluid includes a set of equations, which are usually written as transport equations of main physical parameters—density, velocity, energy, etc. These equations are conservation laws in fluid flows. Traditionally the component form of the equations is usually used, but at the same time, the componentless form (Gibbs approach) could be applied to obtain and transform these equations. In this chapter several main conservation laws are discussed and represented in tensor form, which has many advantages against usually used component form, like simplicity and compactness, independence on reference frames, less errors in transformations, etc. Below we obtain and analyze continuity and momentum equations and vorticity and energy transport equations, and we discuss also about the divergent form of transport Eqs.

2. Continuity equation

Continuity equation is the mass conservation law for a fluid flow and is presented as a scalar equation, which connects density ρ and velocity of fluid particles $\vec{V}$, and for any liquid it could be written as

$$\frac{\partial \rho}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\rho + \rho \vec{\nabla} \cdot \vec{V} = 0, \qquad (1)$$

where $\vec{\nabla} = \vec{e}_i \frac{\partial}{\partial x_i} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_2 \frac{\partial}{\partial x_3}$ is the Hamilton operator and the dot is a symbol of a scalar product.

Hereinafter, the Einstein summation convention is used by default.

It also can be written in two other equivalent forms [1, 2]:

$$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \text{ and } \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V} = 0. \tag{2}$$

In the case of incompressible fluid, we could obtain its simplified form:

$$\vec{\nabla}\cdot\vec{V} = 0. \tag{3}$$

Let us apply gradient operator to continuity equation (Eq. (1)):

$$\vec{\nabla}\frac{\partial\rho}{\partial t} + \vec{\nabla}\left[\left(\vec{V}\cdot\vec{\nabla}\right)\rho\right] + \vec{\nabla}\left[\rho\vec{\nabla}\cdot\vec{V}\right] = 0.$$

As a result, we obtain vector equation:

$$\frac{\partial}{\partial t}\vec{\nabla}\rho + \vec{\nabla}\rho\cdot\vec{\nabla}\vec{V}^{T} + \vec{V}\cdot\vec{\nabla}\vec{\nabla}\rho + \left(\vec{\nabla}\cdot\vec{V}\right)\vec{\nabla}\rho + \rho\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) = 0, \tag{4}$$

which could be written in a more compact form:

$$\frac{d}{dt}\vec{\nabla}\rho + \vec{\nabla}\vec{V}\cdot\vec{\nabla}\rho + \left(\vec{\nabla}\cdot\vec{V}\right)\vec{\nabla}\rho + \rho\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) = 0 \tag{5}$$

or by a little bit different way:

$$\frac{d}{dt}\vec{\nabla}\rho + \vec{\nabla}\vec{V}\cdot\vec{\nabla}\rho + \vec{\nabla}\left(\rho\vec{\nabla}\cdot\vec{V}\right) = 0. \tag{6}$$

These equations contain gradient of vector $\vec{V}$ divergence, which [3] equal to

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) = \Delta\vec{V} + \vec{\nabla}\times\vec{\nabla}\times\vec{V}.$$

For incompressible fluid the left part of this relation is equal to zero; therefore for rotation of velocity vector, we can write:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{V} = -\Delta\vec{V}. \tag{7}$$

In the case of compressible fluid in accordance with Eq. (6), we have additional terms in the right part of the equation:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{V} = -\left(\Delta\vec{V} + \frac{1}{\rho}\frac{d}{dt}\vec{\nabla}\rho + \frac{1}{\rho}\vec{\nabla}\vec{V}\cdot\vec{\nabla}\rho + \frac{\left(\vec{\nabla}\cdot\vec{V}\right)}{\rho}\vec{\nabla}\rho\right). \tag{8}$$

Continuity equation can be also written in tensor form:

$$\frac{1}{3}tr\left(\frac{\partial\rho}{\partial t}\underline{E}\right) + tr\left(\vec{\nabla}\rho\vec{V}\right) = 0. \tag{9}$$

The tensor $\vec{\nabla}\rho\vec{V}$ can be represented as

$$\vec{\nabla}\rho\vec{V} = \vec{e}_k \frac{\partial}{\partial x_k}\rho V_j \vec{e}_j = \frac{\partial}{\partial x_k}\rho V_j \vec{e}_k \vec{e}_j = \\ = \left[\frac{\partial\rho}{\partial x_k}V_j + \rho\frac{\partial V_j}{\partial x_k}\right]\vec{e}_k\vec{e}_j = \vec{\nabla}\rho\otimes\vec{V} + \rho\vec{\nabla}\vec{V}. \tag{10}$$

Finally, continuity equation can be written in form

$$\frac{\partial\rho}{\partial t} + tr\left[\vec{\nabla}\rho\otimes\vec{V} + \rho\vec{\nabla}\vec{V}\right] = 0. \tag{11}$$

Convective derivatives of density and pressure (and any another scalar quantitatives) also can be written in tensor form:

$$\left(\vec{V}\cdot\vec{\nabla}\right)\rho = tr\left(\vec{V}\otimes\vec{\nabla}\rho\right); \quad \left(\vec{V}\cdot\vec{\nabla}\right)p = tr\left(\vec{V}\otimes\vec{\nabla}p\right),$$

i.e., convective derivative is equal to trace of corresponding tensor.

In addition, for the divergence of the product of scalar and vector functions, we can obtain the following relation:

$$\vec{\nabla}\cdot\rho\vec{V} = \left(\vec{\nabla}\rho\otimes\vec{V} + \rho\vec{\nabla}\vec{V}\right) : \underline{E}.$$

3. Equations of motion of fluid with constant and variable properties

The equation of a motion in terms of stress [4, 5] is

$$\rho\frac{d\vec{V}}{dt} = \vec{\nabla}\cdot\underline{\sigma} + \rho\vec{f}, \tag{12}$$

where $\vec{f}$ is the body force per unit mass and $\vec{\nabla}\cdot\underline{\sigma}$ is the divergence of stress tensor $\underline{\sigma}$. In accordance with Newton's law, tensor $\underline{\sigma}$ for an incompressible fluid is

$$\underline{\sigma} = -p\underline{E} + 2\mu\underline{S}, \tag{13}$$

where p is the pressure; μ is the fluid shear (dynamic) viscosity; and $\underline{S} = \frac{1}{2}\left(\vec{\nabla}\vec{V} + \vec{\nabla}\vec{V}^T\right)$ is the rate of strain tensor. Due to relation $\vec{\nabla}\cdot\vec{V} = 0$, when $\mu \neq const$ divergence of stress tensor $\underline{\sigma}$ is written as

$$\vec{\nabla}\cdot\underline{\sigma} = -\vec{\nabla}p + \vec{\nabla}\mu\cdot 2\underline{S} + 2\mu\vec{\nabla}\cdot\underline{S} = -\vec{\nabla}p + \vec{\nabla}\mu\cdot 2\underline{S} + \mu\Delta\vec{V}. \tag{14}$$

Then equation of motion of incompressible fluid (Navier–Stokes equation) at $\mu \neq const$ is

$$\rho\frac{d\vec{V}}{dt} = -\vec{\nabla}p + \mu\Delta\vec{V} + \vec{\nabla}\mu\cdot 2\underline{S} + \rho\vec{f}. \tag{15}$$

Additional term $\vec{\nabla}\mu \cdot 2\underline{S}$ relates to changing of shear viscosity. In Cartesian coordinates Eq. (15) has the form:

$$\rho \frac{dV_i}{dt} = -\frac{\partial p}{\partial x_i} + \mu \Delta V_i + \frac{\partial \mu}{\partial x_j}\left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right) + \rho f_i.$$

In case of compressible fluid with variable viscosity, the equation will contain a term with divergence $\vec{\nabla} \cdot \vec{V}$, which is not equal to zero now. Rheological relation is this case has the form [2]:

$$\underline{\sigma} = -p\underline{E} - \frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\underline{E} + 2\mu\underline{S}. \tag{16}$$

Let us introduce the denotation:

$$p + \frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right) = p'; \tag{17}$$

then we can write Eq. (16) as

$$\underline{\sigma} = -p'\underline{E} + 2\mu\underline{S}. \tag{18}$$

Divergence of this tensor at $\mu \neq const$ is

$$\vec{\nabla} \cdot \underline{\sigma} = -\vec{\nabla}p' + \vec{\nabla}\mu \cdot 2\underline{S} + \mu\Delta\vec{V} + \mu\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right).$$

As a result, equation of motion of compressible fluid with variable viscosity has the form:

$$\rho \frac{d\vec{V}}{dt} = -\vec{\nabla}p' + \mu\Delta\vec{V} + \mu\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) + \vec{\nabla}\mu \cdot 2\underline{S} + \rho\vec{f}, \tag{19}$$

in Cartesian coordinates

$$\rho\left[\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j}\right] = -\frac{\partial p'}{\partial x_i} + \mu\Delta V_i + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial V_j}{\partial x_j}\right) + \frac{\partial \mu}{\partial x_j}\left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right) + \rho f_i.$$

If we represent fluid particle acceleration as the sum of local and convective terms, then (Eq. (19)) will take the form:

$$\rho\frac{\partial\vec{V}}{\partial t} + \rho\left(\vec{V} \cdot \vec{\nabla}\right)\vec{V} = -\vec{\nabla}p' + \mu\Delta\vec{V} + \mu\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) + \vec{\nabla}\mu \cdot 2\underline{S} + \rho\vec{f}, \tag{20}$$

considering viscosity variability is especially important for turbulent flow modeling using the Boussinesq hypothesis with turbulent viscosity μ_t.

Let us apply divergence operation to the Navier–Stokes equation for compressible fluid with variable viscosity. With this purpose we shall apply operation $\left(\vec{\nabla}\cdot\right)$ to each vector term of Eq. (20):

$$\vec{\nabla}\cdot\rho\frac{\partial\vec{V}}{\partial t}=\vec{\nabla}\rho\cdot\frac{\partial\vec{V}}{\partial t}+\rho\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{V};$$

$$\vec{\nabla}\cdot\rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\vec{\nabla}\rho\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]+\rho\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}+\rho\left(\vec{\nabla}\cdot\vec{V}\right)\left(\vec{V}\cdot\vec{\nabla}\right);$$

$$\vec{\nabla}\cdot\left(-\vec{\nabla}p'\right)=-\Delta p';$$

$$\vec{\nabla}\cdot\mu\Delta\vec{V}=\vec{\nabla}\mu\cdot\Delta\vec{V}+\mu\Delta\left(\vec{\nabla}\cdot\vec{V}\right);$$

$$\vec{\nabla}\cdot\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)=\vec{\nabla}\mu\cdot\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)+\mu\Delta\left(\vec{\nabla}\cdot\vec{V}\right);$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\mu\cdot 2\underline{S}\right)=\vec{\nabla}\vec{\nabla}\mu:\vec{\nabla}\vec{V}+\vec{\nabla}\mu\cdot\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)+\vec{\nabla}\vec{\nabla}\mu:\vec{\nabla}\vec{V}^{T}+\vec{\nabla}\mu\cdot\Delta\vec{V}.$$

If the fluid motion occurs in gravity force field, then there is potential $U=gz$, where z is the vertical coordinate and the body force per unit mass is $\vec{f}=-\vec{\nabla}U$. In this case $\vec{\nabla}\cdot\rho\vec{f}=\vec{\nabla}\rho\cdot\vec{f}+\rho\vec{\nabla}\cdot\vec{f}=-\vec{\nabla}\rho\cdot\vec{\nabla}U-\rho\Delta U$.

Function U is linear; therefore $\Delta U=0$ and

$$\vec{\nabla}\cdot\rho\vec{f}=-\vec{\nabla}\rho\cdot\vec{\nabla}U.$$

As a result of applied divergence operation to Navier–Stokes equation at $\mu\neq const$, we obtain scalar equation:

$$\rho\frac{d}{dt}\left(\vec{\nabla}\cdot\vec{V}\right)+\vec{\nabla}\rho\cdot\frac{d\vec{V}}{dt}+\rho\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}=-\Delta p'+\vec{\nabla}2\mu\cdot\left(\Delta V+\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)\right)+$$
$$+2\mu\Delta\left(\vec{\nabla}\cdot\vec{V}\right)+\vec{\nabla}\vec{\nabla}:2\mu\underline{S}-\vec{\nabla}\rho\cdot\vec{\nabla}U. \tag{21}$$

In the case of incompressible fluid, we have

$$\rho\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}=-\Delta p+\vec{\nabla}\cdot 2\mu\Delta\vec{V}+\vec{\nabla}\vec{\nabla}:2\mu\underline{S}; \tag{22}$$

if also $\mu=const$, then

$$\rho\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}=-\Delta p. \tag{23}$$

Now we consider the general case of fluid motion, taking into account its compressibility.

The set of equations of motion of an incompressible fluid contains two—Navier–Stokes and continuity (one vector equation and one scalar equation) [2, 3]:

$$\begin{cases}\rho\dfrac{d\vec{V}}{dt}=-\vec{\nabla}p+\mu\Delta\vec{V}+\rho\vec{f}\\ \vec{\nabla}\cdot\vec{V}=0\end{cases}. \tag{24}$$

This set of two equations is closed: it contains two unknown quantities—velocity vector $\vec{V}$ and pressure per two equations. The set describes laminar flows; in turbulent flows it becomes unclosed because Reynolds stress tensor appeared.

In case of compressible flows at $\rho \neq const$, divergence of velocity is $\vec{\nabla} \cdot \vec{V} \neq 0$, and the Navier–Stokes equation (Eq. (19)) of a fluid motion at $\mu = const$ has the form:

$$\rho \frac{d\vec{V}}{dt} = -\vec{\nabla} p' + \mu \Delta \vec{V} + \mu \vec{\nabla} \left(\vec{\nabla} \cdot \vec{V} \right) + \rho \vec{f}, \tag{25}$$

where p' is defined by Eq. (17). Continuity equation is written in the form of Eq. (1). If $\rho \neq const$, the set of equations (Eq. (25) and Eq. (1)) becomes unclosed, because density will also be unknown. To close the set of equation, energy equation is used, which contains one more unknown scalar quantity—temperature T. To determine temperature T, state equation is used; usually in fluid dynamics, it is the Mendeleev-Clapeyron equation. Energy equation could be written as the equation of specific internal energy transport:

$$\rho \frac{du}{dt} = -\vec{\nabla} \cdot \vec{q} + \underline{\sigma} : \vec{\nabla} \vec{V} + \rho q_s, \tag{26}$$

where u is the specific internal energy (for ideal gas it could be expressed with the help of the isochore heat capacity, $du = c_v dT$); $\vec{q}$ is the heat flux vector (in laminar flow by Fourier's law, $\vec{q} = -\lambda \vec{\nabla} T$); λ is the thermal conductivity of the material; and q_s is the heat flux from internal or external sources.

Mendeleev-Clapeyron equation has the form

$$p = \rho R T, \tag{27}$$

where R is the universal gas constant. In case of real gas or fluid, the state equation becomes more complicated.

For liquids it usually supposes ρ = *const*. This condition is applicable for gas motions and also in case the velocities of gas particles are less than 1/3 of sound velocity.

Eqs. (1), (25), (26), and (27) are valid for laminar regime of motion. In case of turbulent regime in these equations, correlations will appear, caused by velocity, density, and temperature pulsations. For closure of the set of equations of turbulent motion, additional relations are required.

4. Vorticity vector and its associated tensor

Vorticity $\vec{\omega}$ is a vector quantity, which characterizes velocity field:

$$\vec{\omega} = \frac{1}{2} rot \vec{V} = \frac{1}{2} \vec{\nabla} \times \vec{V}, \tag{28}$$

in component form

$$\vec{\omega} = \frac{1}{2} \frac{\partial V_j}{\partial x_i} \varepsilon_{kij} \vec{e}_k,$$

where ε_{ijk} is the Levi-Civita tensor in component form. Components of $\vec{\omega}$ vector are

$$\left.\begin{aligned} \omega_1 = \omega_x = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) \\ \omega_2 = \omega_y = \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) \\ \omega_3 = \omega_z = \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) \end{aligned}\right\}. \tag{29}$$

Vorticity vector $\vec{\omega}$ and spin tensor $\underline{\Omega}$ are connected with each other through Levi-Civita tensor ${}^3\underline{\varepsilon} = \varepsilon_{ijk}\vec{e}_i\vec{e}_j\vec{e}_k$. These quantities are mutually associated. They say that tensor $\underline{\Omega} = \frac{1}{2}\left(\vec{\nabla}\vec{V} - \vec{\nabla}\vec{V}^T\right)$ is associated to vector $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V}$, because the following relations are satisfied:

$$\underline{\Omega} : \left({}^3\underline{\varepsilon}\right) = \left({}^3\underline{\varepsilon}\right) : \underline{\Omega} = -2\vec{\omega} = -\vec{\nabla} \times \vec{V}. \tag{30}$$

And vice versa, vector $\vec{\omega}$ is associated with tensor $\underline{\Omega}$ since the following expression is valid:

$$\left({}^3\underline{\varepsilon}\right) \cdot \vec{\omega} = \vec{\omega} \cdot \left({}^3\underline{\varepsilon}\right) = \underline{\Omega}. \tag{31}$$

In Eq. (30) spin tensor $\underline{\Omega}$ is translated to the vector $\vec{\omega}$; in Eq. (31) vector $\vec{\omega}$ is associated with spin tensor.

Let us prove expression Eq. (30):

$$\begin{aligned} \underline{\Omega} : \left({}^3\underline{\varepsilon}\right) &= \frac{1}{2}\left(\vec{\nabla}\vec{V} - \vec{\nabla}\vec{V}^T\right) : \left({}^3\underline{\varepsilon}\right) = \frac{1}{2}\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right)\vec{e}_i\vec{e}_j : \varepsilon_{pqk}\vec{e}_p\vec{e}_q\vec{e}_k = \\ &= \frac{1}{2}\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right)\varepsilon_{jik}\vec{e}_k = \frac{1}{2}\varepsilon_{ji1}\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right)\vec{e}_1 + \frac{1}{2}\varepsilon_{ji2}\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right)\vec{e}_2 + \\ &+ \frac{1}{2}\varepsilon_{ji3}\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right)\vec{e}_3 = \frac{1}{2}\left(\frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2} - \frac{\partial V_3}{\partial x_2} + \frac{\partial V_2}{\partial x_3}\right)\vec{e}_1 + \\ &+ \frac{1}{2}\left(\frac{\partial V_3}{\partial x_1} - \frac{\partial V_1}{\partial x_3} - \frac{\partial V_1}{\partial x_3} + \frac{\partial V_3}{\partial x_1}\right)\vec{e}_2 + \frac{1}{2}\left(\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1} - \frac{\partial V_2}{\partial x_1} + \frac{\partial V_1}{\partial x_2}\right)\vec{e}_3 = \\ &= -2\omega_1\vec{e}_1 - 2\omega_2\vec{e}_2 - 2\omega_3\vec{e}_3 = -2\vec{\omega} = -\vec{\nabla} \times \vec{V}. \end{aligned}$$

The same for Eq. (31):

$$\left({}^3\underline{\varepsilon}\right) \cdot \vec{\omega} = \varepsilon_{ijk}\vec{e}_i\vec{e}_j\vec{e}_k \cdot \omega_s\vec{e}_s = \omega_k\varepsilon_{ijk}\vec{e}_i\vec{e}_j = \frac{1}{2}\frac{\partial V_s}{\partial x_t}\varepsilon_{kts}\varepsilon_{ijk}\vec{e}_i\vec{e}_j.$$

Let us descry components of this second-rank tensor: when $i = j$ they are equal to zero; when $i = 1, j = 2$ they are

$$\frac{1}{2}\frac{\partial V_s}{\partial x_t}\varepsilon_{kts}\varepsilon_{12k} = \frac{1}{2}\frac{\partial V_s}{\partial x_t}\varepsilon_{3ts} = \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right);$$

values for all $i, j = 1, 2, 3$ could be obtained by the same way.

The matrix of components of this tensor is

$$\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1}-\frac{\partial V_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial V_3}{\partial x_1}-\frac{\partial V_1}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial V_1}{\partial x_2}-\frac{\partial V_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1}-\frac{\partial V_1}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial V_1}{\partial x_3}-\frac{\partial V_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial V_2}{\partial x_3}-\frac{\partial V_3}{\partial x_2}\right) & 0 \end{pmatrix}.$$

It can be seen that it is a matrix of components of the antisymmetric tensor $\underline{\Omega}$, which means that relation (31) is valid.

It is easy to see also that

$$\vec{\omega}\cdot\underline{\Omega}=0, \vec{\omega}\times\underline{\Omega}=0, \vec{\nabla}\cdot\vec{\omega}=0. \tag{32}$$

5. Vorticity transport equation

Rotation of convective acceleration of a fluid particle could be written as

$$\vec{\nabla}\times\left[\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=2\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}-2\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}. \tag{33}$$

Evidence of this equation can be performed by writing convective acceleration according to the formula:

$$\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\vec{\nabla}\frac{\vec{V}^2}{2}-\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right). \tag{34}$$

Gradient of vorticity is the pseudo tensor of rank 2:

$$\vec{\nabla}\vec{\omega}=\vec{\nabla}\left(\frac{1}{2}\vec{\nabla}\times\vec{V}\right)=\vec{e}_s\frac{\partial}{\partial x_s}\left(\frac{1}{2}\frac{\partial V_j}{\partial x_i}\varepsilon_{kij}\vec{e}_k\right)=\frac{1}{2}\frac{\partial^2 V_j}{\partial x_s\partial x_i}\varepsilon_{kij}\vec{e}_s\otimes\vec{e}_k. \tag{35}$$

Trace of this tensor is $tr\vec{\nabla}\vec{\omega}=0$. It is also possible to distinguish the symmetric and antisymmetric parts of this tensor:

$$\frac{1}{4}\left(\frac{\partial^2 V_j}{\partial x_s\partial x_i}\varepsilon_{kij}+\frac{\partial^2 V_j}{\partial x_k\partial x_i}\varepsilon_{sij}\right)\vec{e}_s\otimes\vec{e}_k, \frac{1}{4}\left(\frac{\partial^2 V_j}{\partial x_s\partial x_i}\varepsilon_{kij}-\frac{\partial^2 V_j}{\partial x_k\partial x_i}\varepsilon_{sij}\right)\vec{e}_s\otimes\vec{e}_k.$$

Let us assume that the fluid is incompressible, $\mu = const$, and its motion occurs in the field of potential mass forces. In this case with the help of Eq. (34), we can obtain the Navier–Stokes equation in the form:

$$\frac{\partial\vec{V}}{\partial t}+\vec{\nabla}\left(\frac{V^2}{2}+P+U\right)+\left(\vec{\nabla}\times\vec{V}\right)\times\vec{V}=\nu\Delta\vec{V};$$

now we apply curl operation ($rot=\vec{\nabla}\times$) to the left and right parts of this equation:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{V}\right)+\vec{\nabla}\times\left[\left(\vec{\nabla}\times\vec{V}\right)\times\vec{V}\right]=\nu\Delta\left(\vec{\nabla}\times\vec{V}\right).$$

We could rewrite the second term of the left part in this form:

$$\left(\vec{V}\cdot\vec{\nabla}\right)\left(\vec{\nabla}\times\vec{V}\right)-\left[\left(\vec{\nabla}\times\vec{V}\right)\cdot\vec{\nabla}\right]\vec{V}+\left(\vec{\nabla}\times\vec{V}\right)\left(\vec{\nabla}\cdot\vec{V}\right)-\vec{V}\left(\vec{\nabla}\cdot 2\vec{\omega}\right),$$

but $\vec{\nabla}\cdot\vec{\omega}=0$, and fluid is incompressible ($\vec{\nabla}\cdot\vec{V}=0$); therefore, this term finally can be written as

$$2\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}-2\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}.$$

As a result, we can obtain transport equation of vortices in an incompressible viscous fluid, which is named as the generalized Helmholtz equation:

$$\frac{\partial\vec{\omega}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}-\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}=\nu\Delta\vec{\omega}, \tag{36}$$

or in more compact form:

$$\frac{d\vec{\omega}}{dt}-\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}=\nu\Delta\vec{\omega}. \tag{37}$$

It is necessary to note that in the case of compressible fluid and at $\mu\neq const$, this equation becomes much more complicated.

This transport equation can be written in another form, considering the equality $\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}-\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}=\vec{\nabla}\times\left(\vec{\omega}\times\vec{V}\right)$. Then

$$\frac{\partial\vec{\omega}}{\partial t}+\vec{\nabla}\times\left(\vec{\omega}\times\vec{V}\right)=\nu\Delta\vec{\omega}. \tag{38}$$

If we apply divergence operation to Eq. (36), then for incompressible fluid we obtain

$$\vec{\nabla}\cdot\left[\frac{\partial\vec{\omega}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}-\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}\right]=\vec{\nabla}\cdot\nu\Delta\vec{\omega};$$

because $\vec{\nabla}\cdot\vec{\omega}=0$, we have

$$\vec{\nabla}\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}\right]-\vec{\nabla}\left[\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}\right]=0.$$

On the other hand,

$$\vec{\nabla}\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}\right]=\vec{\nabla}\vec{V}:\vec{\nabla}\vec{\omega}+\left(\vec{V}\cdot\vec{\nabla}\right)\left(\vec{\nabla}\cdot\vec{\omega}\right);\ \vec{\nabla}\cdot\left[\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}\right]$$
$$=\vec{\nabla}\vec{V}:\vec{\nabla}\vec{\omega}+\left(\vec{\omega}\cdot\vec{\nabla}\right)\left(\vec{\nabla}\cdot\vec{V}\right),$$

and, finally, we have $0=0$, i.e., we shall not obtain a new expression.

For the second power of vorticity, we can write

$$\omega^2=\vec{\nabla}\vec{V}^{T}:\vec{\nabla}\vec{V}-\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}, \tag{39}$$

and now, if we scalar multiply transport equation of vortices by $\vec{\omega}$:

$$\vec{\omega}\cdot\left[\frac{\partial\vec{\omega}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{\omega}-\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{V}\right]=\vec{\omega}\cdot\nu\Delta\vec{\omega};$$

then we obtain scalar transport equation of ω^2:

$$\frac{d\omega^2}{dt}-\left(\vec{\omega}\otimes\vec{\omega}\right):\vec{\nabla}\vec{V}=\nu\Delta\omega^2-\nu\vec{\nabla}\vec{\omega}^T:\vec{\nabla}\vec{\omega}. \tag{40}$$

For incompressible fluid $\vec{\nabla}\cdot\underline{S}=\vec{\nabla}\cdot\underline{\Omega}$ because $\vec{\nabla}\cdot\underline{S}=\frac{1}{2}\Delta\vec{V}+\frac{1}{2}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)$ and $\vec{\nabla}\cdot\underline{\Omega}=\frac{1}{2}\Delta\vec{V}-\frac{1}{2}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)$.

As we already know $\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=-2\vec{V}\cdot\underline{\Omega}$; therefore, Eq. (34) can be written as

$$\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\vec{\nabla}\frac{V^2}{2}+2\vec{V}\cdot\underline{\Omega}. \tag{41}$$

Let us write one more equation:

$$\vec{\nabla}\cdot\left[\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\right]=-\vec{\nabla}\cdot\left[\vec{V}\cdot2\underline{\Omega}\right]=\vec{\nabla}\cdot\left[\vec{V}\cdot\vec{\nabla}\vec{V}^T-\vec{V}\cdot\vec{\nabla}\vec{V}\right]=$$

$$=\vec{\nabla}\cdot\left[V_i\frac{\partial V_i}{\partial x_j}-V_i\frac{\partial V_j}{\partial x_i}\right]\vec{e_j}=\frac{\partial}{\partial x_j}\left[V_i\frac{\partial V_i}{\partial x_j}-V_i\frac{\partial V_j}{\partial x_i}\right]=\frac{\partial V_i}{\partial x_j}\frac{\partial V_i}{\partial x_j}+V_i\frac{\partial^2 V_i}{\partial x_j\partial x_j}-$$

$$-\frac{\partial V_i}{\partial x_j}\frac{\partial V_j}{\partial x_i}-V_i\frac{\partial^2 V_j}{\partial x_i\partial x_j}=\vec{\nabla}\vec{V}^T:\vec{\nabla}\vec{V}-\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}+\vec{V}\cdot\Delta\vec{V}-\vec{V}\cdot\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right).$$

Therefore,

$$\vec{\nabla}\cdot\left[\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\right]=\vec{\nabla}\vec{V}^T:\vec{\nabla}\vec{V}-\vec{\nabla}\vec{V}:\vec{\nabla}\vec{V}+\vec{V}\cdot\left[\Delta\vec{V}-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)\right] \tag{42}$$

This equation also can be written in the next form:

$$\vec{\nabla}\cdot\left[\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\right]=2\underline{\Omega}:\vec{\nabla}\vec{V}+2\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\Omega}\right)$$

or

$$\vec{\nabla}\cdot\left[\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\right]=2\underline{\Omega}:\underline{\Omega}+2\underline{\Omega}:\underline{S}+2\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\Omega}\right).$$

One more interesting relation is

$$\vec{V}\cdot\vec{\nabla}\vec{V}=\vec{V}\cdot\underline{\Omega}+\vec{V}\cdot\underline{S},$$

but as we already mentioned, $\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=2\vec{V}\cdot\underline{\Omega}$; therefore,

$$\vec{V}\cdot\underline{S}=\vec{\nabla}\frac{V^2}{2}-\frac{1}{2}\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right). \tag{43}$$

The vector product of gradients of scalar functions gives us a vector; in terms of rotation of a vector function, we can write

$$\vec{\nabla} f \times \vec{\nabla}\varphi = \vec{\nabla} \times \left(f\vec{\nabla}\varphi\right). \tag{44}$$

Really, the left part of this equation is a vector:

$$\vec{\nabla} f \times \vec{\nabla}\varphi = \left(\frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_2}\right)\vec{e}_1 + \left(\frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_3}\right)\vec{e}_2 + \left(\frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_1}\right)\vec{e}_3,$$

and the right part is.

$$\begin{aligned}
\vec{\nabla} \times \left(f\vec{\nabla}\varphi\right) &= \vec{e}_i \frac{\partial}{\partial x_i} \times f \frac{\partial \varphi}{\partial x_j}\vec{e}_j = \left(\vec{e}_i \times \vec{e}_j\right)\frac{\partial}{\partial x_i} f \frac{\partial \varphi}{\partial x_j} = \varepsilon_{kij}\frac{\partial}{\partial x_i}\left(f\frac{\partial \varphi}{\partial x_j}\right)\vec{e}_k = \\
&= \varepsilon_{1ij}\frac{\partial}{\partial x_i}\left(f\frac{\partial \varphi}{\partial x_j}\right)\vec{e}_1 + \varepsilon_{2ij}\frac{\partial}{\partial x_i}\left(f\frac{\partial \varphi}{\partial x_j}\right)\vec{e}_2 + \varepsilon_{3ij}\frac{\partial}{\partial x_i}\left(f\frac{\partial \varphi}{\partial x_j}\right)\vec{e}_3 = \\
&= \left(\frac{\partial}{\partial x_2} f \frac{\partial \varphi}{\partial x_3} - \frac{\partial}{\partial x_3} f \frac{\partial \varphi}{\partial x_2}\right)\vec{e}_1 + \left(\frac{\partial}{\partial x_3} f \frac{\partial \varphi}{\partial x_1} - \frac{\partial}{\partial x_1} f \frac{\partial \varphi}{\partial x_3}\right)\vec{e}_2 \\
&\quad + \left(\frac{\partial}{\partial x_1} f \frac{\partial \varphi}{\partial x_2} - \frac{\partial}{\partial x_2} f \frac{\partial \varphi}{\partial x_1}\right)\vec{e}_3 = \\
&= \left(\frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_2}\right)\vec{e}_1 + \left(\frac{\partial f}{\partial x_3}\frac{\partial \varphi}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_3}\right)\vec{e}_2 + \left(\frac{\partial f}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial \varphi}{\partial x_1}\right)\vec{e}_3.
\end{aligned}$$

Therefore, Eq. (44) is valid.
The vector product of gradients of scalar functions also can be written as

$$\vec{\nabla} f \times \vec{\nabla}\varphi = \frac{1}{2}\left(\vec{\nabla}\varphi \otimes \vec{\nabla} f - \vec{\nabla} f \otimes \vec{\nabla}\varphi\right) : \left({}^3\underline{\varepsilon}\right) = -\vec{\nabla} \cdot \left[\left(f\vec{\nabla}\varphi\right) \cdot \left({}^3\underline{\varepsilon}\right)\right]. \tag{45}$$

Here we have in component form:

$$\begin{aligned}
\vec{\nabla} \cdot \left[\left(f\vec{\nabla}\varphi\right) \cdot {}^3\underline{\varepsilon}\right] &= \vec{\nabla} \cdot \left[f\frac{\partial \varphi}{\partial x_s}\vec{e}_s \cdot \varepsilon_{kij}\vec{e}_k\vec{e}_i\vec{e}_j\right] = \vec{\nabla} \cdot \left[f\frac{\partial \varphi}{\partial x_k}\varepsilon_{kij}\vec{e}_i\vec{e}_j\right] = \\
&= \vec{e}_t\frac{\partial}{\partial x_t} \cdot f\frac{\partial \varphi}{\partial x_k}\varepsilon_{kij}\vec{e}_i\vec{e}_j = \frac{\partial}{\partial x_i}\left(f\frac{\partial \varphi}{\partial x_k}\right)\varepsilon_{kij}\vec{e}_j = \\
&= \left(\frac{\partial}{\partial x_i} f \frac{\partial \varphi}{\partial x_k}\right)\varepsilon_{ki1}\vec{e}_1 + \left(\frac{\partial}{\partial x_i} f \frac{\partial \varphi}{\partial x_k}\right)\varepsilon_{ki2}\vec{e}_2 + \left(\frac{\partial}{\partial x_i} f \frac{\partial \varphi}{\partial x_k}\right)\varepsilon_{ki3}\vec{e}_3 = \\
&= \left(\frac{\partial f}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial \varphi}{\partial x_3}\right)\vec{e}_1 + \left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial \varphi}{\partial x_3} - \frac{\partial f}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_1}\right)\vec{e}_2 \\
&\quad + \left(\frac{\partial f}{\partial x_2} \cdot \frac{\partial \varphi}{\partial x_1} - \frac{\partial f}{\partial x_1} \cdot \frac{\partial \varphi}{\partial x_2}\right)\vec{e}_3.
\end{aligned}$$

It is easy to see that this is equal to expression for $\vec{\nabla} f \times \vec{\nabla}\varphi$ with the minus sign.

6. Mechanical energy equation

Mechanical energy balance equation can be obtained as a scalar product of each member of Eq. (12) on velocity vector $\vec{V}$:

$$\vec{V}\cdot\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)+\vec{V}\cdot\rho\vec{f}. \tag{46}$$

Transformations of the left part lead us to the following results:

$$\vec{V}\cdot\frac{\partial\vec{V}}{\partial t}=V_k\vec{e}_k\cdot\frac{\partial}{\partial t}V_i\vec{e}_i=V_k\frac{\partial V_i}{\partial t}\left(\vec{e}_k\cdot\vec{e}_i\right)=V_k\frac{\partial V_i}{\partial t}\delta_{ki}=V_k\frac{\partial V_k}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}V_kV_k=\frac{1}{2}\frac{\partial V^2}{\partial t};$$

$$V\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=V_k\vec{e}_k\cdot V_j\frac{\partial}{\partial x_j}V_i\vec{e}_i=V_kV_j\frac{\partial V_i}{\partial x_j}\left(\vec{e}_k\cdot\vec{e}_i\right)=$$
$$=V_kV_j\frac{\partial V_i}{\partial x_j}\delta_{ki}=V_kV_j\frac{\partial V_k}{\partial x_j}=V_j\frac{\partial}{\partial x_j}\frac{1}{2}V_kV_k=\left(\vec{V}\cdot\vec{\nabla}\right)\frac{V^2}{2}.$$

It is easy to see that in sum the left part is the material derivative of kinetic energy of a fluid particle—quantity $\frac{d}{dt}\frac{V^2}{2}$. Then Eq. (46) takes the form:

$$\rho\frac{d}{dt}\left[\frac{V^2}{2}\right]=\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)+\vec{V}\cdot\rho\vec{f}. \tag{47}$$

Usually stress tensor is defined as the sum $\underline{\sigma}=-p\underline{E}+\underline{\tau}$, where $\underline{\tau}$ is the shear stress tensor. Then the first term of the right part of Eq. (46) takes the form:

$$\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)=-\vec{V}\cdot\left(\vec{\nabla}\cdot p\underline{E}\right)+\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\tau}\right).$$

Now we can represent equation of mechanical energy balance (Eq. (46)) considering $\vec{\nabla}\cdot p\underline{E}=\vec{\nabla}p$ as follows:

$$\rho\frac{d}{dt}\left[\frac{V^2}{2}\right]=-\vec{V}\cdot\vec{\nabla}p+\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\tau}\right)+\rho\vec{V}\cdot\vec{f}. \tag{48}$$

The first member of the right part of Eq. (46) is power of stresses $\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)$, which can be written in the form:

$$\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)=\vec{\nabla}\cdot\left(\vec{V}\cdot\underline{\sigma}\right)-\underline{\sigma}:\vec{\nabla}\vec{V}. \tag{49}$$

It is easy to be proven if we rewrite this expression in component form in Cartesian coordinates. In this case the left part of Eq. (49) is

$$\vec{V}\cdot\left(\vec{\nabla}\cdot\underline{\sigma}\right)=V_k\vec{e}_k\cdot\left(\vec{e}_i\frac{\partial}{\partial x_i}\cdot\sigma_{sj}\vec{e}_s\vec{e}_j\right)=V_k\vec{e}_k\cdot\left[\frac{\partial\sigma_{sj}}{\partial x_i}\left(\vec{e}_i\cdot\vec{e}_s\right)\vec{e}_j\right]=$$
$$=V_k\vec{e}_k\cdot\frac{\partial\sigma_{ij}}{\partial x_i}\vec{e}_j=V_k\frac{\partial\sigma_{ij}}{\partial x_i}\left(\vec{e}_k\cdot\vec{e}_j\right)=V_k\frac{\partial\sigma_{ik}}{\partial x_i}.$$

The first term of the right part of Eq. (49) in component form is

$$\vec{\nabla}\cdot\left(\vec{V}\cdot\underline{\sigma}\right)=\vec{\nabla}\cdot\left(V_s\vec{e}_s\cdot\sigma_{ij}\vec{e}_i\vec{e}_j\right)=\vec{\nabla}\cdot\left[V_s\sigma_{ij}\left(\vec{e}_s\cdot\vec{e}_i\right)\vec{e}_j\right]=\vec{\nabla}\cdot\left(V_i\sigma_{ij}\vec{e}_j\right)=$$
$$=\vec{e}_k\frac{\partial}{\partial x_k}\cdot V_i\sigma_{ij}\vec{e}_j=\delta_{kj}\frac{\partial}{\partial x_k}V_i\sigma_{ij}=\frac{\partial}{\partial x_k}V_i\sigma_{ik}=\frac{\partial V_i}{\partial x_k}\sigma_{ik}+V_i\frac{\partial\sigma_{ik}}{\partial x_k}.$$

The second term of the right part of Eq. (49) in component form is

$$\sigma : \vec{\nabla}\vec{V} = \sigma_{ij}\vec{e}_i\vec{e}_j : \frac{\partial V_s}{\partial x_k}\vec{e}_k\vec{e}_s = \sigma_{ij}\frac{\partial V_s}{\partial x_k}\left(\vec{e}_j \cdot \vec{e}_k\right)\left(\vec{e}_i \cdot \vec{e}_s\right) =$$

$$= \sigma_{ij}\frac{\partial V_s}{\partial x_k}\delta_{jk}\delta_{is} = \sigma_{ij}\frac{\partial V_i}{\partial x_j} = \frac{\partial V_i}{\partial x_k}\sigma_{ik}.$$

Finally Eq. (49) in component form is

$$V_k\frac{\partial \sigma_{ik}}{\partial x_i} = \frac{\partial V_i}{\partial x_k}\sigma_{ik} + V_i\frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial V_i}{\partial x_k}\sigma_{ik}.$$

Due to the symmetry of the stress tensor $\underline{\sigma} = \underline{\sigma}^T$, this expression is an identity (i.e., the left side is equal to the right one). Indeed, the second term of the right-hand side of this relation, after re-designating the index i by k and vice versa, takes the form $V_k\frac{\partial \sigma_{ki}}{\partial x_i}$, but $\sigma_{ki} = \sigma_{ik}$; therefore both parts of the expression are equal to each other. Thus, equality (49) is valid.

We could simplify the last term of the right part of Eq. (46) if we introduce potential U of mass forces in field of gravity (z axis is positive upwards) as earlier $\vec{f} = -\vec{\nabla}U$, $U = gz = \vec{r}\cdot\vec{g}$, where $\vec{g}$ is the gravity acceleration vector in the field of gravity. Then

$$\vec{V}\cdot\rho\vec{f} = -\rho\vec{V}\cdot\vec{\nabla}U = -\rho\frac{d\vec{r}}{dt}\cdot\left(\vec{\nabla}U\right) = -\rho\frac{1}{dt}\left(\vec{\nabla}U\cdot d\vec{r}\right) = -\rho\frac{dU}{dt}. \qquad (50)$$

After substituting the above expressions into Eq. (46), we obtain the equation for mechanical energy of a fluid flow:

$$\rho\frac{d}{dt}\left(\frac{V^2}{2} + U\right) = \vec{\nabla}\cdot\left(\vec{V}\cdot\underline{\sigma}\right) - \underline{\sigma} : \vec{\nabla}\vec{V}. \qquad (51)$$

In the left part of this equation, we observe the total mechanical energy of a fluid flow as the sum of kinetic and potential energy of the flow [6]. Often the right part of Eq. (51) is written in another form, where stress tensor is written as the sum $\sigma = -p\underline{E} + \underline{\tau}$. Then the right part of Eq. (51) will have the form:

$$-\vec{\nabla}\cdot\left(\vec{V}\cdot p\underline{E}\right) + \vec{\nabla}\cdot\left(\vec{V}\cdot\underline{\tau}\right) + p\underline{E} : \vec{\nabla}\vec{V} - \underline{\tau} : \vec{\nabla}\vec{V}. \qquad (52)$$

The first member is

$$-\vec{\nabla}\cdot\left(\vec{V}\cdot p\underline{E}\right) = -\vec{\nabla}\cdot\left(V_i\vec{e}_i\cdot p\delta_{jk}\vec{e}_j\vec{e}_k\right) = -\vec{\nabla}\cdot\left(\vec{V}_i p\delta_{jk}\delta_{ij}\vec{e}_k\right) =$$

$$= -\vec{\nabla}\cdot\left(pV_k\vec{e}_k\right) = -\vec{e}_s\frac{\partial}{\partial x_s}\cdot pV_k\vec{e}_k = -\frac{\partial pV_k}{\partial x_s}\delta_{sk} = -\frac{\partial}{\partial x_k}pV_k =$$

$$= -V_k\frac{\partial p}{\partial x_k} - p\frac{\partial V_k}{\partial x_k} = -\left(\vec{V}\cdot\vec{\nabla}\right)p - p\vec{\nabla}\cdot\vec{V}.$$

The third member is

$$p\underline{E} : \vec{\nabla}\vec{V} = p\delta_{ij}\vec{e}_i\vec{e}_j : \frac{\partial V_s}{\partial x_k}\vec{e}_k\vec{e}_s = p\frac{\partial V_s}{\partial x_k}\delta_{ij}\delta_{jk}\delta_{is} = p\frac{\partial V_k}{\partial x_k} = p\vec{\nabla}\cdot\vec{V}.$$

When we substitute these terms in Eq. (52) and then in Eq. (51), the equation for mechanical energy of a fluid flow will take the form:

$$\rho \frac{d}{dt}\left(\frac{V^2}{2} + U\right) = -\left(\vec{V} \cdot \vec{\nabla}\right)p + \vec{\nabla}\left(\vec{V} \cdot \underline{\tau}\right) - \tau : \vec{\nabla}\vec{V}. \tag{53}$$

As a result, we could conclude that the rate of change of total mechanical energy of a flow is equal to the sum of the powers of the pressure forces and viscous friction.

Navier–Stokes equation for a steady flow of viscous incompressible fluid is

$$\rho\left(\vec{V} \cdot \vec{\nabla}\right)\vec{V} = -\vec{\nabla}p + \mu\Delta\vec{V} + \rho\vec{f}. \tag{54}$$

The Laplacian of velocity in the right part can be written in the form:

$$\Delta\vec{V} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right). \tag{55}$$

It can be obtained by consideration of operation $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right)$:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{e}_i \frac{\partial}{\partial x_i} \times \left(\frac{\partial V_j}{\partial x_k}\varepsilon_{skj}\vec{e}_s\right) =$$

$$= \frac{\partial^2 V_j}{\partial x_i \partial x_k}\varepsilon_{skj}\varepsilon_{lis}\vec{e}_l = \frac{\partial^2 V_j}{\partial x_i \partial x_k}\varepsilon_{skj}\varepsilon_{1is}\vec{e}_1 + \frac{\partial^2 V_j}{\partial x_i \partial x_k}\varepsilon_{skj}\varepsilon_{2is}\vec{e}_2 + \frac{\partial^2 V_j}{\partial x_i \partial x_k}\varepsilon_{skj}\varepsilon_{3is}\vec{e}_3.$$

The member with the basis vector $\vec{e}_1$ is determined as

$$\frac{\partial^2 V_j}{\partial x_i \partial x_k}\varepsilon_{skj}\varepsilon_{1is} = \frac{\partial^2 V_j}{\partial x_2 \partial x_k}\varepsilon_{3kj} - \frac{\partial^2 V_j}{\partial x_3}\varepsilon_{2kj} = \frac{\partial^2 V_2}{\partial x_2 \partial x_1} - \frac{\partial^2 V_1}{\partial x_3 \partial x_3} + \frac{\partial^2 V_3}{\partial x_3 \partial x_1} =$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial^2 V_1}{\partial x_1} + \frac{\partial^2 V_2}{\partial x_2} + \frac{\partial^2 V_3}{\partial x_3}\right) - \frac{\partial^2 V_1}{\partial x_1^2} - \frac{\partial^2 V_1}{\partial x_2^2} - \frac{\partial^2 V_1}{\partial x_3^2} = \frac{\partial}{\partial x_1}\left(\vec{\nabla} \cdot \vec{V}\right) - \Delta V_1.$$

The same can be written for the members with basis vectors $\vec{e}_2$ and $\vec{e}_3$. As a result, we obtain

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \left[\frac{\partial}{\partial x_1}\left(\vec{\nabla} \cdot \vec{V}\right) - \Delta V_1\right]\vec{e}_1 + \left[\frac{\partial}{\partial x_2}\left(\vec{\nabla} \cdot \vec{V}\right) - \Delta V_2\right]\vec{e}_2 +$$

$$+ \left[\frac{\partial}{\partial x_3}\left(\vec{\nabla} \cdot \vec{V}\right) - \Delta V_3\right]\vec{e}_3 = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \Delta\vec{V}.$$

And therefore formula Eq. (55) is valid.

One more useful expression based on Eq. (55) is

$$\vec{\nabla} \cdot \underline{S} = \Delta\vec{V} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) = 2\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \vec{\nabla} \times \vec{\nabla} \times \vec{V}.$$

Using Eqs. (34) and (55) and considering the mass force in field of gravity with the help of potential U, $U = -gz$ (z axis is positive upwards), we obtain instead Eq. (54):

$$\vec{\nabla}\frac{V^2}{2}-\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=-\frac{1}{\rho}\vec{\nabla}p+\nu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)-\nu\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right)+\vec{\nabla}U,$$

where $\nu=\frac{\mu}{\rho}$ is the kinematic (momentum) viscosity of fluid.

In incompressible fluid $\rho=const$, $\vec{\nabla}\cdot\vec{V}=0$ and then we have

$$\vec{\nabla}\left(\frac{V^2}{2}+\frac{p}{\rho}+gz\right)=\underbrace{\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)-\nu\vec{\nabla}\times\vec{\nabla}\times\vec{V}}_{\left(\vec{V}-\nu\vec{\nabla}\right)\times\vec{\nabla}\times\vec{V}}. \tag{56}$$

The gradient of total mechanical energy of a fluid particle $E=\frac{V^2}{2}+\frac{p}{\rho}+gz$ depends on vortex structure of the flow. When $\nabla\times\vec{V}=0$ the right part of Eq. (56) is zero, and then $E=const$ in the whole area of the flow.

It is possible to obtain the divergence form of Eq. (54) for incompressible fluid considering $\vec{\nabla}\cdot\vec{V}=0$ and using Eq. (55) and the relation:

$$\vec{\nabla}\cdot\left(\vec{V}\otimes\vec{V}\right)=\vec{V}\left(\vec{\nabla}\cdot\vec{V}\right)+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}. \tag{57}$$

Then Eq. (54) will take the form:

$$\vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right)=-\vec{\nabla}p+\vec{\nabla}\rho U-\mu\vec{\nabla}\times\vec{\nabla}\times\vec{V}. \tag{58}$$

Using concepts of identity tensor $\underline{I}$ and Levi-Civita tensor ${}^3\underline{\varepsilon}=\varepsilon_{ijk}\vec{e}_i\vec{e}_j\vec{e}_k$, we can write members of the right part in the divergence form, and as a result, the whole Navier–Stokes equation for steady flow of incompressible fluid can be written as

$$\vec{\nabla}\cdot\left[\rho\vec{V}\otimes\vec{V}+p\underline{I}-\rho U\underline{I}+\rho\nu\left({}^3\varepsilon\right)\cdot\left(\vec{\nabla}\times\vec{V}\right)\right]=0. \tag{59}$$

The last term of Eq. (59) can be considered in a more simple form due to relation for incompressible fluid $\Delta\vec{V}=\vec{\nabla}\cdot\vec{\nabla}\vec{V}=\vec{\nabla}\cdot\underline{S}$. Finally, the divergence form of the Navier–Stokes equation for steady flow of an incompressible fluid is

$$\vec{\nabla}\cdot\left[\rho\left(\vec{V}\otimes\vec{V}\right)+(p+\rho gz)\underline{I}-\mu\underline{S}\right]=0. \tag{60}$$

7. Energy equation for moving fluid

The first law of thermodynamics connects internal energy, heat, and work. In the case of moving fluid, it can be written as follows:

$$\rho\frac{du}{dt}=-\vec{\nabla}\cdot\vec{q}+\underline{\sigma}:\vec{\nabla}\vec{V}+q_v, \tag{61}$$

where t is the time; u is the specific internal energy; $\vec{q}$ is the heat flux density vector due to thermal conductivity; $\underline{\sigma}$ is the stress tensor; and q_v is the value of heat entering into the particle volume from action of external or internal sources per unit time. In this expression, the colon denotes double scalar product of tensors; in this case these are the stress tensor and velocity gradient tensor.

The physical meaning of this equation is that the rate of change of internal energy per unit volume is equal to rate of energy supply due to heat conduction, due to dissipation of mechanical energy of the flow, and due to heat from external or internal sources. Since stress tensor $\underline{\sigma}$ can be written as $\underline{\sigma} = -p\underline{E} + \underline{\tau}$, where $\underline{\tau}$ is the shear stress tensor, taking into account material derivative definition, we can rewrite Eq. (61) in this form:

$$\rho\left[\frac{\partial u}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)u\right] = -\vec{\nabla}\cdot\vec{q} - p\vec{\nabla}\cdot\vec{V} + \underline{\tau} : \vec{\nabla}\vec{V} + q_v. \tag{62}$$

This is the energy equation in terms of transfer of specific internal energy u.

Vector $\vec{q}$ in the energy equation is determined by Fourier's law:

$$\vec{q} = -\lambda\vec{\nabla}T, \tag{63}$$

where T is the temperature and λ is the coefficient of thermal conductivity.

Fourier's law of thermal conductivity can also be written in terms of enthalpy, which for an ideal gas is related to temperature by the formula $h = c_p T$, where c_p is the isobar heat capacity. Then considering $\lambda = \rho c_p a$, where a is the thermal diffusivity, heat flux density vector can be written in the form:

$$\vec{q} = \lambda\vec{\nabla}T = \rho c_p a\vec{\nabla}T = \rho\frac{\nu}{a/\nu}\vec{\nabla}c_p T = \rho\frac{\nu}{\mathrm{Pr}}\vec{\nabla}h = \frac{\mu}{\mathrm{Pr}}\vec{\nabla}h,$$

where $\mathrm{Pr} = a/\nu$ is the Prandtl number.

In Cartesian coordinates Eq. (62) can be written as follows:

$$\rho\left[\frac{\partial u}{\partial t} + V_j\frac{\partial u}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\lambda\frac{\partial T}{\partial x_j} - p\frac{\partial V_j}{\partial x_j} + \tau_{ij}\frac{\partial V_i}{\partial x_j} + q_v. \tag{64}$$

The terms $p\vec{\nabla}\cdot\vec{V}$ and $\underline{\tau} : \vec{\nabla}\vec{V}$ show us in a moving fluid heating or cooling can occur. The term $p\vec{\nabla}\cdot\vec{V}$ may cause significant change of temperature, when gas expands (compresses) rapidly. The term $\underline{\tau} : \vec{\nabla}\vec{V}$ is always positive; it characterizes dissipation of mechanical energy and its transformation to heat energy. This scalar quantity is usually named as Rayleigh dissipation function [6] and denoted as $\underline{\tau} : \vec{\nabla}\vec{V} = \Phi$. Let us write this function in Cartesian coordinates for Newtonian viscous fluid, when rheological relation has the form:

$$\underline{\tau} = -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\underline{E} + 2\mu\underline{S}, \tag{65}$$

where μ is the fluid shear viscosity and $\underline{S}$ is the strain rate tensor.

Now we could write the dissipative term $\underline{\tau} : \vec{\nabla}\vec{V}$ in Eq. (62) by simple transformations:

$$\Phi = \underline{\tau} : \vec{\nabla}\vec{V} = \left(-\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\underline{E} + 2\mu\underline{S}\right) : \vec{\nabla}\vec{V} =$$

$$= -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\underline{E} : \vec{\nabla}\vec{V} + 2\mu\cdot\frac{1}{2}\left(\vec{\nabla}\vec{V} + \vec{\nabla}\vec{V}^{T}\right) : \vec{\nabla}\vec{V} =$$

$$= -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\delta_{ij}\vec{e}_i\vec{e}_j : \frac{\partial V_s}{\partial x_k}\vec{e}_k\vec{e}_s + \mu\left[\vec{\nabla}\vec{V} : \vec{\nabla}\vec{V} + \vec{\nabla}\vec{V}^{T} : \vec{\nabla}\vec{V}\right] =$$

$$= -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\delta_{ij}\delta_{jk}\delta_{is}\frac{\partial V_s}{\partial x_k} + \mu\left[\frac{\partial V_j}{\partial x_i}\vec{e}_i\vec{e}_j : \frac{\partial V_s}{\partial x_k}\vec{e}_k\vec{e}_s + \frac{\partial V_i}{\partial x_j}\vec{e}_i\vec{e}_j : \frac{\partial V_s}{\partial x_k}\vec{e}_k\vec{e}_s\right] =$$

$$= -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\frac{\partial V_i}{\partial x_i} + \mu\left[\frac{\partial V_j}{\partial x_i}\frac{\partial V_s}{\partial x_k}\delta_{jk}\delta_{is} + \frac{\partial V_i}{\partial x_j}\frac{\partial V_s}{\partial x_k}\delta_{jk}\delta_{is}\right] = -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)^2 +$$

$$+\mu\left[\frac{\partial V_j}{\partial x_i}\frac{\partial V_i}{\partial x_j} + \frac{\partial V_i}{\partial x_j}\frac{\partial V_i}{\partial x_j}\right] = -\frac{2}{3}\mu\left(\frac{\partial V_i}{\partial x_i}\right)^2 + \mu\left[\frac{\partial V_j}{\partial x_i}\frac{\partial V_i}{\partial x_j} + \frac{\partial V_i}{\partial x_j}\frac{\partial V_i}{\partial x_j}\right].$$

Thus, in component form Rayleigh function Φ can be written as

$$\Phi = \underline{\tau} : \vec{\nabla}\vec{V} = -\frac{2}{3}\mu\left(\frac{\partial V_i}{\partial x_i}\right)^2 + \mu\left[\frac{\partial V_j}{\partial x_i}\frac{\partial V_i}{\partial x_j} + \frac{\partial V_i}{\partial x_j} : \frac{\partial V_i}{\partial x_j}\right], \tag{66}$$

or in usual notations

$$\Phi = \mu\left\{-\frac{2}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)^2 + 2\left[\left(\frac{\partial V_x}{\partial x}\right)^2 + \left(\frac{\partial V_y}{\partial y}\right)^2 + \left(\frac{\partial V_z}{\partial z}\right)^2\right]\right.$$
$$\left. + \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)^2 + \left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)^2 + \left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)^2\right\}. \tag{67}$$

This function can also be written in the componentless form:

$$\Phi = \underline{\tau} : \vec{\nabla}\vec{V} = -\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)^2 + \mu\left[\vec{\nabla}\vec{V} : \vec{\nabla}\vec{V} + \vec{\nabla}\vec{V} : \vec{\nabla}\vec{V}^{T}\right]. \tag{68}$$

For perfect gases [6, 7] internal energy is connected with temperature by the relation $du = c_v dT$, where c_v is the isochore thermal capacity. Then, instead of Eq. (62) with the help of expression for vector $\vec{q}$, we can write equation for temperature transport in the form:

$$\rho c_v\left[\frac{\partial T}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)T\right] = \vec{\nabla}\cdot\left(\lambda\vec{\nabla}T\right) - p\left(\vec{V}\cdot\vec{\nabla}\right) + \underline{\tau} : \vec{\nabla}\vec{V} + q_v. \tag{69}$$

The energy equation (Eq. (62)) can be also written in terms of enthalpy $h = u + \frac{p}{\rho}$. With this purpose we need to add the term $\rho\frac{d}{dt}\left(\frac{p}{\rho}\right)$ to the left and right parts of the equation. Then we obtain in the left part $\rho\frac{dh}{dt}$, but in the right part, we shall get this term in transformed form shown as follows:

$$\rho\frac{d}{dt}\left(\frac{p}{\rho}\right) = \frac{dp}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = \frac{dp}{dt} + \rho\left(\vec{\nabla}\cdot\vec{V}\right). \tag{70}$$

Here we also used continuity equation (Eq. (2)). Finally, we can obtain energy equation in the form of enthalpy transport as

$$\rho \frac{dh}{dt} = -\vec{\nabla} \cdot \vec{q} + \frac{dp}{dt} + \underline{\tau} : \vec{\nabla}\vec{V} + q_v. \tag{71}$$

This is the second form of energy equation for the perfect gas in which $dh = c_p dT$, where c_p is the isobar heat capacity, which leads to the following transport equation for temperature:

$$\rho c_p \frac{dT}{dt} = \vec{\nabla} \cdot \left(\lambda \vec{\nabla} T\right) + \frac{dp}{dt} + \underline{\tau} : \vec{\nabla}\vec{V} + q_v. \tag{72}$$

One more form of the energy equation can be written if we introduce stagnation enthalpy $h + \frac{V^2}{2}$. To do it we need to add equations for mechanical energy (Eq. (48) to Eq. (71)); as a result we obtain the equation:

$$\rho \frac{d}{dt}\left(h + \frac{V^2}{2}\right) = -\vec{\nabla} \cdot \vec{q} + \frac{\partial p}{\partial t} + \vec{\nabla} \cdot \left(\underline{\tau} \cdot \vec{V}\right) + \rho \vec{V} \cdot \vec{f} + q_v. \tag{73}$$

Here we used the relation:

$$\vec{\nabla} \cdot \left(\underline{\tau} \cdot \vec{V}\right) = \vec{V} \cdot \left(\vec{\nabla} \cdot \underline{\tau}\right) + \underline{\tau} : \vec{\nabla}\vec{V}, \tag{74}$$

which is easy to be proven if we write it down in the component form considering symmetry of stress tensor $\underline{\tau}$.

As a result, the dissipative term in Eq. (73) can be written as follows:

$$\vec{\nabla} \cdot \left(\underline{\tau} \cdot \vec{V}\right) = \Psi + \Phi, \tag{75}$$

where Φ is the Rayleigh dissipation function and $\Psi = \vec{V} \cdot \left(\vec{\nabla} \cdot \underline{\tau}\right)-$ is the scalar quantity, which can be named as additional dissipation function. This additional dissipation function in Cartesian coordinates for Newtonian Stokes liquid can be written as

$$\begin{aligned}
\Psi &= \vec{V} \cdot \left(\vec{\nabla} \cdot \underline{\tau}\right) = \vec{V} \cdot \left\{\vec{\nabla} \cdot \left[-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\underline{E} + 2\mu\underline{S}\right]\right\} = \\
&= \vec{V} \cdot \left\{\vec{e}_k \frac{\partial}{\partial x_k} \cdot \left[-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\delta_{ij}\vec{e}_i\vec{e}_j + 2\mu S_{ij}\vec{e}_i\vec{e}_j\right]\right\} = \\
&= \vec{V} \cdot \left\{\frac{\partial}{\partial x_k} \cdot \left[-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\delta_{ij}\delta_{ki}\vec{e}_j + 2\mu S_{ij}\delta_{ki}\vec{e}_j\right]\right\} = \\
&= \vec{V} \cdot \left\{\frac{\partial}{\partial x_k} \cdot \left[-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\vec{e}_k + 2\mu S_{kj}\vec{e}_j\right]\right\} = \\
&= V_s \vec{e}_s \cdot \left\{\frac{\partial}{\partial x_k}\left[-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\vec{e}_k + 2\mu S_{kj}\vec{e}_j\right]\right\} = \\
&= \left(\vec{V} \cdot \vec{\nabla}\right)\left(-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\right) + V_j \frac{\partial}{\partial x_k}\left(2\mu S_{kj}\right).
\end{aligned}$$

In usual axis designations, the first term is

$$\left(V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}+V_z\frac{\partial}{\partial z}\right)\left(-\frac{2}{3}\mu\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)\right),$$

while second term is

$$\begin{aligned}V_j\frac{\partial}{\partial x_k}\left(2\mu S_{kj}\right)&=V_1\frac{\partial}{\partial x_k}(2\mu S_{k1})+V_2\frac{\partial}{\partial x_k}(2\mu S_{k2})+V_3\frac{\partial}{\partial x_k}(2\mu S_{k3})=\\&=V_1\left[\frac{\partial}{\partial x_1}(2\mu S_{11})+\frac{\partial}{\partial x_2}(2\mu S_{21})+\frac{\partial}{\partial x_3}(2\mu S_{31})\right]+\\&\quad+V_2\left[\frac{\partial}{\partial x_1}(2\mu S_{12})+\frac{\partial}{\partial x_2}(2\mu S_{22})+\frac{\partial}{\partial x_3}(2\mu S_{32})\right]+\\&\quad+V_3\left[\frac{\partial}{\partial x_1}(2\mu S_{13})+\frac{\partial}{\partial x_2}(2\mu S_{23})+\frac{\partial}{\partial x_3}(2\mu S_{33})\right].\end{aligned}$$

Finally function Ψ in Cartesian coordinates can be written as follows:

$$\begin{aligned}\Psi=&-\frac{2}{3}\left[V_x\frac{\partial}{\partial x}\mu\left(\vec{\nabla}\cdot\vec{V}\right)+V_y\frac{\partial}{\partial y}\mu\left(\vec{\nabla}\cdot\vec{V}\right)+V_z\frac{\partial}{\partial z}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\right]\\&+V_x\left[\frac{\partial}{\partial x}\left(2\mu\frac{\partial V_x}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial V_z}{\partial x}+\frac{\partial V_x}{\partial z}\right)\right]\\&+V_y\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial V_y}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial V_z}{\partial y}+\frac{\partial V_y}{\partial z}\right)\right]\\&+V_z\left[\frac{\partial}{\partial x}\mu\left(\frac{\partial V_z}{\partial x}+\frac{\partial V_x}{\partial z}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial V_z}{\partial y}+\frac{\partial V_y}{\partial z}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial V_z}{\partial z}\right)\right].\end{aligned}\tag{76}$$

The fourth form of the energy equation can be written in terms of entropy s transport. According to the fundamental thermodynamic relation,

$$Tds=dh-\frac{1}{\rho}dp.\tag{77}$$

Hence we have

$$T\frac{ds}{dt}=\frac{dh}{dt}-\frac{1}{\rho}\frac{dp}{dt},$$

and then if we substitute quantity $\frac{dh}{dt}$ from Eq. (77) to Eq. (71), we obtain

$$\rho T\frac{ds}{dt}=-\vec{\nabla}\cdot\vec{q}+\underline{\tau}:\vec{\nabla}\vec{V}+q_v.\tag{78}$$

All forms of the equation energy (in terms of internal energy, enthalpy, stagnation enthalpy, and entropy) are equivalent.

Equation for temperature field of an arbitrary gas in the form of equation of transport of temperature T can be obtained from Eq. (62) or Eq. (71). In these cases quantities $\frac{du}{dt}$ and $\frac{dh}{dt}$ for arbitrary gases and liquids must be specified using known formulas, which follow from Maxwell's relations:

$$du = c_v dT - \frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho - p\right]d\rho, \tag{79}$$

$$dh = c_p dT + \frac{1}{\rho^2}\left[\rho + T\left(\frac{\partial \rho}{\partial T}\right)_p\right]dp. \tag{80}$$

The subscripts in derivatives here fix the parameters, with the constancy of which the derivatives are calculated. From these formulas the expressions for derivatives can be obtained:

$$\frac{du}{dt} = c_v\frac{dT}{dt} - \frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho - p\right]\frac{d\rho}{dt}, \tag{81}$$

$$\frac{dh}{dt} = c_p\frac{dT}{dt} + \frac{1}{\rho^2}\left[\rho + T\left(\frac{\partial \rho}{\partial T}\right)_p\right]\frac{dp}{dt}. \tag{82}$$

If we substitute them into Eq. (62) and Eq. (71), we obtain two forms of the equation energy in terms of temperature transport.

In heat transfer problems, boundary conditions are specified in three different kinds—the first, second, and third kind:

1. The boundary conditions of the first kind consist in setting the temperature on the surface of the body.

2. The boundary conditions of the second kind are setting of the distribution of the heat flux density q on the surface of the body.

3. The boundary conditions of the third kind consist in setting the temperature of the flow over the surface of the body and the heat transfer conditions on its surface.

8. Divergence form of transport equations

Material derivative of any physical quantity Θ multiplied by density ρ always can be written in the "divergent" form as

$$\rho\frac{d\Theta}{dt} = \frac{\partial}{\partial t}\rho\Theta + \vec{\nabla}\cdot\left(\rho\vec{V}\Theta\right). \tag{83}$$

This directly follows from the continuity equation [Eq. (2)].

Let us consider in detail the following cases for three ranks of a certain physical quantity Θ.

8.1 Quantity Θ is a scalar

Let us assume that quantity Θ is temperature *T:*

$$\rho\frac{dT}{dt} = \frac{\partial}{\partial t}\rho T + \vec{\nabla}\cdot\left(\rho\vec{V}T\right). \tag{84}$$

We could prove this equality if we write the left and right parts in component form. For the left part, we have

$$\rho\frac{dT}{dt}=\frac{\partial T}{\partial t}+\rho\left(\vec{V}\cdot\vec{\nabla}\right)T=\rho\frac{\partial T}{\partial t}+\rho V_j\frac{\partial T}{\partial x_j}.$$

For the right part, we have

$$\frac{\partial}{\partial t}\rho T+\vec{\nabla}\cdot\left(\rho\vec{V}T\right)=T\frac{\partial\rho}{\partial t}+\rho\frac{\partial T}{\partial t}+\frac{\partial}{\partial x_j}\rho V_j T=$$

$$=T\frac{\partial\rho}{\partial t}+\rho\frac{\partial T}{\partial t}+TV_j\frac{\partial\rho}{\partial x_j}+\rho T\frac{\partial V_j}{\partial x_j}+\rho V_j\frac{\partial T}{\partial x_j}=$$

$$=\underbrace{T\left(\frac{\partial\rho}{\partial t}+V_j\frac{\partial\rho}{\partial x_j}+\rho\frac{\partial V_j}{\partial x_j}\right)}_{\frac{\partial\rho}{\partial t}+\vec{V}\cdot\vec{\nabla}\rho+\rho\vec{\nabla}\cdot\vec{V}=0}+\rho\frac{\partial T}{\partial t}+\rho V_j\frac{\partial T}{\partial x_j}=\rho\frac{\partial T}{\partial t}+\rho V_j\frac{\partial T}{\partial x_j}.$$

Since the expression in parentheses is zero (due to continuity equation), the equality of the left and right parts is obvious.

8.2 Quantity Θ is a vector

Let us assume that quantity Θ is velocity $\vec{V}$. In this case its material derivative can be written in the form:

$$\rho\frac{d\vec{V}}{dt}=\frac{\partial}{\partial t}\rho\vec{V}+\vec{\nabla}\cdot\left(\rho\vec{V}\vec{V}\right).\tag{85}$$

Here in the last term, we see tensor $\rho\vec{V}\otimes\vec{V}$ (momentum flow tensor); the sign of tensor multiplication $\otimes$ is omitted for ease of recording.

We could prove this equality if we write the left and right parts in the component form and use continuity equation.

For the left part, we have

$$\rho\frac{d\vec{V}}{dt}=\rho\frac{\partial V}{\partial t}+\rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\rho\frac{dV_i}{dt}\vec{e}_i+\rho V_j\frac{\partial V_i}{\partial x_j}\vec{e}_i.$$

The first term of the right part is

$$\frac{\partial}{\partial t}\rho\vec{V}=\frac{\partial}{\partial t}\rho V_i\vec{e}_i=\frac{\partial\rho}{\partial t}V_i\vec{e}_i+\rho\frac{\partial V_i}{\partial t}\vec{e}_i.$$

The second term of the right part is

$$\vec{\nabla}\cdot\left(\rho\vec{V}\vec{V}\right)=\vec{e}_k\frac{\partial}{\partial x_k}\cdot\rho V_jV_i\vec{e}_j\vec{e}_i=\frac{\partial}{\partial x_k}\rho V_jV\left(\vec{e}_k\cdot\vec{e}_j\right)\vec{e}_i=$$

$$=\delta_{kj}\frac{\partial}{\partial x_k}\rho V_jV_i\vec{e}_i=\frac{\partial}{\partial x_j}\rho V_jV_i\vec{e}_i=\rho\frac{\partial V_j}{\partial x_j}V_i\vec{e}_i+V_j\frac{\partial\rho}{\partial x_j}V_i\vec{e}_i+\rho V_j\frac{\partial V_i}{\partial x_j}\vec{e}_i.$$

The right part as a whole is

$$\rho\frac{\partial V_i}{\partial t}\vec{e}_i + \rho V_j\frac{\partial V_i}{\partial x_j}\vec{e}_i + V_i\vec{e}_i\underbrace{\left(\frac{\partial\rho}{\partial t} + V_j\frac{\partial\rho}{\partial x_j} + \rho\frac{\partial V_j}{\partial x_j}\right)}_{\frac{\partial\rho}{\partial t}+\vec{V}\cdot\vec{\nabla}\rho+\rho\vec{\nabla}\cdot\vec{V}=0}.$$

Therefore, the expression (Eq. (83)) is valid in case Θ is a vector.

8.3 Quantity Θ is a tensor

Let us assume that Θ is a tensor, for instance, stress tensor $\underline{\sigma}$. Stress tensor is a second-rank symmetric tensor, which in Cartesian coordinates can be written in the form $\underline{\sigma} = \sigma_{ij}\vec{e}_i\vec{e}_j$. Let us prove the equality:

$$\rho\frac{d\underline{\sigma}}{dt} = \frac{\partial}{\partial t}\rho\underline{\sigma} + \vec{\nabla}\cdot\left(\rho\vec{V}\underline{\sigma}\right). \tag{86}$$

The left part of this equation in the component form can be written as follows:

$$\rho\frac{d\underline{\sigma}}{dt} = \rho\frac{d}{dt}\sigma_{ij}\vec{e}_i\vec{e}_j = \rho\frac{\partial}{\partial t}\sigma_{ij}\vec{e}_i\vec{e}_j + \rho\left(\vec{V}\cdot\vec{\nabla}\right)\sigma_{ij}\vec{e}_i\vec{e}_j =$$
$$= \rho\frac{\partial\sigma_{ij}}{\partial t}\vec{e}_i\vec{e}_j + \rho V_k\frac{\partial\sigma_{ij}}{\partial x_k}\vec{e}_i\vec{e}_j.$$

The right part is

$$\frac{\partial}{\partial t}\rho\underline{\sigma} + \vec{\nabla}\cdot\left(\rho\vec{V}\underline{\sigma}\right) = \frac{\partial}{\partial t}\rho\sigma_{ij}\vec{e}_i\vec{e}_j + \vec{e}_k\frac{\partial}{\partial x_k}\cdot\rho V_s\sigma_{ij}\vec{e}_s\vec{e}_i\vec{e}_j =$$
$$= \sigma_{ij}\frac{\partial\rho}{\partial t}\vec{e}_i\vec{e}_j + \rho\frac{\partial\sigma_{ij}}{\partial t}\vec{e}_i\vec{e}_j + \delta_{ks}\frac{\partial}{\partial x_k}\rho V_s\sigma_{ij}\vec{e}_i\vec{e}_j =$$
$$= \sigma_{ij}\frac{\partial\rho}{\partial t}\vec{e}_i\vec{e}_j + \rho\frac{\partial\sigma_{ij}}{\partial t}\vec{e}_i\vec{e}_j + \frac{\partial}{\partial x_k}\rho V_k\sigma_{ij}\vec{e}_i\vec{e}_j =$$
$$= \sigma_{ij}\frac{\partial\rho}{\partial t}\vec{e}_i\vec{e}_j + \rho\frac{\partial\sigma_{ij}}{\partial t}\vec{e}_i\vec{e}_j + V_k\sigma_{ij}\frac{\partial\rho}{\partial x_k}\vec{e}_i\vec{e}_j + \rho\sigma_{ij}\frac{\partial V_k}{\partial x_k}\vec{e}_i\vec{e}_j + \rho V_k\frac{\partial\sigma_{ij}}{\partial x_k}\vec{e}_i\vec{e}_j =$$
$$= \rho\frac{\partial\sigma_{ij}}{\partial t}\vec{e}_i\vec{e}_j + \rho V_k\frac{\partial\sigma_{ij}}{\partial x_k}\vec{e}_i\vec{e}_j + \sigma_{ij}\vec{e}_i\vec{e}_j\underbrace{\left(\frac{\partial\rho}{\partial t} + V_k\frac{\partial\rho}{\partial x_k} + \rho\frac{\partial V_k}{\partial x_k}\right)}_{=0}$$

Therefore, the expression (Eq. (83)) is also valid in case Θ is a tensor.

It is necessary to note the derivative $\frac{d}{dt}\underline{\sigma}$, which contains local and convective parts $\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)$, can, at first glance, be used in fluid models when writing the defining equation in the form of a differential transport Equation [2, 5]. However, careful analysis shows that the material derivative for a second-rank tensor is not an invariant quantity [5, 8, 9]. By this reason, instead of derivative $\frac{d}{dt}$, derivative $\frac{D}{Dt}$ is usually used as the material derivative for a second-rank tensor, which contains also rotational part (deviatoric stress rate), which provides symmetry relative to rotations. The rotational part cannot be written in divergent form.

There are different forms of deviatoric stress rate for an arbitrary second-rank tensor $\underline{A}$, for instance:

Jaumann G.: $\underline{A} \cdot \underline{\Omega} + \left(\underline{A} \cdot \underline{\Omega}\right)^T$, $\underline{\Omega}$—antisymmetric spin tensor.

Rivlin R.: $\underline{A} \cdot \vec{\nabla}\vec{V} + \vec{\nabla}\vec{V}^T \cdot \underline{A}$.

Truesdell C.: $\vec{\nabla}\vec{V} \cdot \underline{A} + \underline{A} \cdot \left(\vec{\nabla}\vec{V}\right)^T + \left(tr\vec{\nabla}\vec{V}\right)\underline{A}$.

Oldroyd J., Sedov L.I., etc. [8, 10, 11]

At present, the question of which derivative is more appropriate to use when constructing rheological equations is unclear. The most common is the rotational derivative by Gustav Jaumann. The corresponding material derivative in the form by Jaumann, for an arbitrary tensor of the second rank, has the form

$$\frac{D\underline{A}}{Dt} = \frac{\partial \underline{A}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\underline{A} + \underline{A} \cdot \underline{\Omega} + \left(\underline{A} \cdot \underline{\Omega}\right)^T. \tag{87}$$

Material derivative in the form by Rivlin is written as follows:

$$\frac{D\underline{A}}{Dt} = \frac{\partial \underline{A}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\underline{A} + \underline{A} \cdot \vec{\nabla}\vec{V} + \left(\underline{A} \cdot \vec{\nabla}\vec{V}\right)^T. \tag{88}$$

It is easy to see that Rivlin's derivative differs from Jaumann's one by the additional term $\underline{A} \cdot \underline{S} + \left(\underline{A} \cdot \underline{S}\right)^T$, which is neutral by itself.

Rotational derivative of a symmetric tensor is also a symmetric tensor. As an example, let us consider the rotational derivative of strain-rate tensor and spin tensor:

$$\underline{S} \cdot \underline{\Omega} + \left(\underline{S} \cdot \underline{\Omega}\right)^T = \frac{1}{4}\left[\left(\vec{\nabla}\vec{V} + \vec{\nabla}\vec{V}^T\right) \cdot \left(\vec{\nabla}\vec{V} - \vec{\nabla}\vec{V}^T\right) + \left(\vec{\nabla}\vec{V}^T + \vec{\nabla}\vec{V}\right) \cdot \left(\vec{\nabla}\vec{V}^T - \vec{\nabla}\vec{V}\right)\right] =$$
$$= \frac{1}{2}\left[\vec{\nabla}\vec{V}^T \cdot \vec{\nabla}\vec{V} - \vec{\nabla}\vec{V} \cdot \vec{\nabla}\vec{V}^T\right].$$

As a result, we have obtained the symmetrical second-rank tensor.

9. Conclusions

In this chapter, some applications of tensor calculus in fluid dynamics and heat transfer are presented. Typical transformations of equations and governing relations are discussed. Main conservation equations are given and analyzed. The governing equations of fluid motion and energy were obtained.

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